SPECTRA

Univariate Series

For all *t*, the series X_t can be represented by

$$X_t = a_0^x + \sum_{K=1}^{q} \left(a_K^x \cos 2\pi f_K (t-1) + b_K^x \sin 2\pi f_K (t-1) \right)$$

where

$$t = 1, 2, \ldots, N$$

$$a_0^x = \overline{X},\ \overline{X} = \sum_{t=1}^{N} X_t / N$$

$$a_K^x = \frac{2}{N} \left[\sum_{t=1}^{N} \left(X_t \cos 2\pi f_K (t-1) \right) \right]$$

$$b_K^x = \frac{2}{N} \left[\sum_{t=1}^{N} \left(X_t \sin 2\pi f_K (t-1) \right) \right]$$

$$f_K = K/N$$

$$q = \begin{cases} N/2, & \text{if } N \text{ is even} \\ (N-1)/2, & \text{if } N \text{ is odd} \end{cases}$$

The following statistics are calculated:

Frequency

$$f_K = K/N,\ K = 1,\ldots,q$$

Period

$$1/f_K = N/K,\ K = 1,\ldots,q$$

Fourier Cosine Coefficient

$$a_K^x,\ K = 1,\ldots,q$$

Fourier Sine Coefficient

$$b_K^x = \left(a_K^x - ib_K^x\right)\left(a_K^x + ib_K^x\right)$$

Periodogram

$$l_K^x = \left[\left(a_K^x\right)^2 + \left(b_K^x\right)^2\right]N/2,\ K = 1,\ldots,q$$

spectral density estimate

$$s_K^x = \sum_{j=p}^{p} w_j l_{K+j}^x,\ \text{ where } 2p+1 = m \text{ (number of spans)}$$

and

$$l_{-K}^{x} = l_{K}^{x},\ K = 1,\ldots,q$$
$$l_{0}^{x} = l_{1}^{x}$$
$$l_{K}^{x} = l_{N+1-K} \text{ for } K > q$$

$w_{-p},\ w_{-p+1},\ldots,w_0,\ w_1,\ldots,w_p$ are the periodogram weights defined by different data windows.

Bivariate Series

For the bivariate series X_t and Y_t

$$X_t = a_0^x + \sum_{K=1}^{q}\left(a_K^x \cos 2\pi f_K t + b_K^x \sin 2\pi f_K t\right) \qquad t = 1,\ldots,N$$

$$Y_t = a_0^y + \sum_{K=1}^{q}\left(a_K^y \cos 2\pi f_K t + b_K^y \sin 2\pi f_K t\right)$$

Cross-Periodogram of X and Y

$$\begin{aligned} l_K^{xy} &= \frac{N}{2}\left(a_K^x - i b_K^x\right)\left(a_K^y + i b_K^y\right) \\ &= \frac{N}{2}\left\{\left(a_K^x a_K^y + b_K^x b_K^y\right) + i\left(a_K^x b_K^y - b_K^x a_K^y\right)\right\} \end{aligned}$$

Real $\left(l_K^{xy}\right)$

$$(RC)_K = \frac{N}{2}\left(a_K^x a_K^y + b_K^x b_K^y\right)$$

Imaginary $\left(I_K^{xy}\right)$

$$(IC)_K = \frac{N}{2}\left(a_K^x b_K^y - b_K^x a_K^y\right)$$

Cospectral Density Estimate

$$C_K = \sum_{j=-p}^{p} w_j (RC)_{K+j}$$

Quadrature Spectrum Estimate

$$Q_K = \sum_{j=-p}^{p} w_j (IC)_{K+j}$$

Cross-amplitude Values

$$A_K = \left(Q_K^2 + C_K^2\right)^{1/2}$$

Squared Coherency Values

$$K_K = \frac{A_K^2}{s_K^x s_K^y}$$

Gain Values

$$G_K = \begin{cases} A_K / s_K^x & \text{(gain of } Y_t \text{ over } X_t \text{ at } f_K) \\ A_K / s_K^y & \text{(gain of } X_t \text{ over } Y_t \text{ at } f_K) \end{cases}$$

Phase Spectrum Estimate

$$\Psi_K = \begin{cases} \tan^{-1}(Q_K/C_K) & \text{if } \begin{matrix} Q_K > 0,\ C_K > 0 \\ Q_K < 0,\ C_K > 0 \end{matrix} \\ \tan^{-1}(Q_K/C_K) + \pi & \text{if } Q_K > 0,\ C_K < 0 \\ \tan^{-1}(Q_K/C_K) - \pi & \text{if } Q_K < 0,\ C_K < 0 \end{cases}$$

Data Windows[1]

The following spectral windows can be specified. Each formula defines the upper half of the window. The lower half is symmetric with the upper half. In all formulas, p is the integer part of the number of spans divided by 2. To be concise, the formulas are expressed in terms of the Fejer kernel:

$$F_q(\theta) = \begin{cases} q & \theta = 0, \pm 2\pi, \pm 4\pi, \ldots \\ \frac{1}{q}\left(\frac{\sin(q\theta/2)}{\sin(\theta/2)}\right)^2 & \text{otherwise} \end{cases}$$

and the Dirichlet kernel:

$$D_q(\theta) = \begin{cases} 2q+1 & \theta = 0, \pm 2\pi, \pm 4\pi, \ldots \\ \frac{\sin((2q+1)\theta/2)}{\sin(\theta/2)} & \text{otherwise} \end{cases}$$

where q is any positive real number.

[1] This algorithm applies to SPSS 6.0 and later releases.

HAMMING

Tukey-Hamming window. The weights are

$$W_k = 0.54 D_p(2\pi f_k) + 0.23 D_p\left(2\pi f_k + \frac{\pi}{p}\right) + 0.23 D_p\left(2\pi f_k - \frac{\pi}{p}\right)$$

for $k = 0, \ldots, p$.

TUKEY

Tukey-Hanning window. The weights are

$$W_k = 0.5 D_p(2\pi f_k) + 0.25 D_p\left(2\pi f_k + \frac{\pi}{p}\right) + 0.25 D_p\left(2\pi f_k - \frac{\pi}{p}\right)$$

for $k = 0, \ldots, p$.

PARZEN

Parzen window. The weights are

$$W_k = \frac{1}{p}\left(2 + \cos(2\pi f_k)\right)\left(F_{p/2}(2\pi f_k)\right)^2$$

for $k = 0, \ldots, p$.

BARTLETT

Bartlett window. The weights are

$$W_k = F_p(2\pi f_k)$$

for $k = 0, \ldots, p$.

DANIELL UNIT

Daniell window or rectangular window. The weights are

$$W_k = 1$$

for $k = 0, \ldots, p$.

NONE

No smoothing. If NONE is specified, the spectral density estimate is the same as the periodogram. It is also the case when the number of span is 1.

$$W_{-p}, \ldots, W_0, \ldots, W_p$$

User-specified weights. If the number of weights is odd, the middle weight is applied to the periodogram value being smoothed and the weights on either side are applied to preceding and following values. If the number of weights are even (it is assumed that W_p is not supplied), the weight after the middle applies to the periodogram value being smoothed. It is required that the weight W_0 must be positive.

References

Bloomfield, P. 1976. *Fourier analysis of time series*, New York: John Wiley & Sons, Inc.

Fuller, W. A. 1976. *Introduction to statistical time series*. New York: John Wiley & Sons, Inc.